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UNIVERSALITY OF THE OPERATOR PRODUCT EXPANSIONS OF SCFT₄

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Abstract

We study the operator product algebra of the supercurrent $J_{\alpha\dot{\alpha}}$ and Konishi superfield K in four-dimensional supersymmetric gauge theories. The Konishi superfield appears in the JJ OPE and the algebra is characterized by two central charges c and c' and an anomalous dimension h for K . In free field (one-loop) approximation, $c \sim 3N_v + N_\chi$ and $c' \sim N_\chi$, where N_v and N_χ are, respectively, the number of vector and chiral multiplets in the theory. In higher order c , c' and h depend on the gauge and Yukawa couplings and we obtain the two-loop contributions by combining earlier work on c with our own calculations of c' and h . The major result is that the radiative corrections to the central charges cancel when the one-loop beta-functions vanish, suggesting that c and c' are invariant under continuous deformations of superconformal theories. The behavior of c and c' along renormalization group flows is studied from the viewpoint of a c -theorem.

Introduction.

In the past few years the study of superconformal quantum field theories in four spacetime dimensions (SCFT₄) has received renewed attention. Examples of superconformal invariant theories such as N=4 super Yang-Mills theory and certain N=2 and N=1 supersymmetric gauge theories with suitably chosen matter content [1] have been known for some time. Asymptotically free supersymmetric gauge theories can flow to interacting superconformal theories in the infrared and the infrared fixed points can contain important information about nonperturbative dynamics [2]. Furthermore, there have been studies of the renormalization group flows from the viewpoint of a possible *c*-theorem [3] in four dimensions [4, 5].

In two-dimensional conformal field theories the operator products of the stress tensor and conserved currents reflect the quantum properties of the conformal symmetry [6]. A study of the analogous OPE's in four dimensional N=1 supersymmetric theories was initiated in [7] and a very striking difference with respect to the two-dimensional case was emphasized. One considers the supercurrent superfield $J_{\alpha\dot{\alpha}}(z)$ ($z = x, \theta, \bar{\theta}$) which contains the stress tensor $T_{\mu\nu}(x)$, the supercurrent $S_\mu(x)$ and the R-symmetry current $R_\mu(x)$ as its physical components. The lowest dimensional operator that appears in the $J_{\alpha\dot{\alpha}}(z) J_{\beta\dot{\beta}}(z')$ OPE is a scalar superfield $\Sigma(z)$ of dimension $2 + h$, where h is an anomalous dimension depending on the coupling constants of the theory. As we discuss in more detail below, $\Sigma(z)$ is related to the Konishi superfield $K(z) = \bar{\Phi} e^V \Phi$ (Φ and V are the chiral and gauge superfields of the theory), whose physical components are $K(x) = \bar{\phi}(x)\phi(x)$, an axial current $K_\mu(x) = \frac{1}{2}\bar{\psi}\gamma_\mu\gamma_5\psi + \bar{\phi}\overleftrightarrow{D}_\mu\phi$, and the matter kinetic Lagrangian density. In a schematic notation the OPE's are expected to take the form [7]

$$\begin{aligned} J(z)J(z') &= \frac{c}{(s\bar{s})^3} + \frac{\Sigma(z')}{(s\bar{s})^{2-h/2}} + \cdots, \\ J(z)\Sigma(z') &= \frac{h\Sigma(z')}{(s\bar{s})^{3/2}} + [D, \bar{D}]\frac{\Sigma(z')}{s\bar{s}} + \cdots, \\ \Sigma(z)\Sigma(z') &= \frac{c'}{(s\bar{s})^{2+h}} + \frac{\Sigma(z')}{(s\bar{s})^{1+h/2}} + \cdots \end{aligned} \tag{1}$$

where $s = x - x' + \theta$ -terms is a (chiral) superspace interval and D, \bar{D} are spinor derivatives.

The three numbers c , c' and h characterize a superconformal theory in four dimensions. The central charge c is related to the gravitational trace anomaly of the theory. In a free field theory with N_1 , $N_{1/2}$ and N_0 real or Majorana fields of spin 1, 1/2 and 0 respectively, one-loop calculations give [8]

$$c = \frac{1}{120} (12N_1 + 3N_{1/2} + N_0). \tag{2}$$

In a supersymmetric gauge theory with $N_v = \dim G$ vector multiplets, G denoting the gauge group, and $N_\chi = \dim T$ chiral multiplets in the representation T , one can rewrite this as

$$c = \frac{1}{24} (3N_v + N_\chi). \tag{3}$$

In a free field theory the operator $\Sigma(z)$ is just $\bar{\Phi}\Phi$ and one has $h = 0$ and [7]

$$c' = N_\chi, \quad (4)$$

Thus c is a measure of the total number of degrees of freedom of the theory, while c' measures the number of chiral matter multiplets.

Quantum corrections.

We will study lowest order quantum corrections to c , c' and h , in interacting $N=1$ supersymmetric gauge theories with gauge group G and gauge coupling g . Chiral superfields Φ^i are assigned to the representation T^a of G , in general reducible, and there is a cubic superpotential $W = \frac{1}{6}Y_{ijk}\Phi^i\Phi^j\Phi^k$. Most of our computations were carried out in components of the vector multiplet (in Wess-Zumino gauge) $V^a(z) \rightarrow A_\mu^a(x)$, $\lambda^a(x)$, and the scalar multiplet $\Phi^i(z) \rightarrow \phi^i(x)$, $\psi^i(x)$, with auxiliary fields eliminated. Using the formalism of Euclidean Majorana spinors of Nicolai [9] the action reads

$$\begin{aligned} S = \int d^4x \left[\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}\bar{\lambda}\not{D}\lambda + \overline{D_\mu\phi}D_\mu\phi + \frac{1}{2}\bar{\psi}\not{D}\psi \right. \\ \left. + i\sqrt{2}g(\bar{\lambda}^a\bar{\phi}_iT^{ai}_jL\psi^j - \bar{\psi}_iRT^{ai}_j\phi^j\lambda^a) \right. \\ \left. - \frac{1}{2}(\bar{\psi}^iLY_{ijk}\phi^k\psi^j + \bar{\psi}_iR\bar{Y}^{ijk}\bar{\phi}_k\psi_j) \right. \\ \left. + \frac{1}{2}g^2(\bar{\phi}_iT^{ai}_j\phi^j)^2 + \frac{1}{4}Y_{ijk}\bar{Y}^{ilm}\phi^j\phi^k\bar{\phi}_l\bar{\phi}_m \right] \end{aligned} \quad (5)$$

where L and R are projection operators $L, R = \frac{1}{2}(1 \mp \gamma_5)$. Note that the index i is both a flavor and color index.

The theory contains the classically conserved, but anomalous, R -current

$$R_\mu(x) = \frac{1}{2}\bar{\lambda}\gamma_\mu\gamma_5\lambda - \frac{1}{6}\bar{\psi}\gamma_\mu\gamma_5\psi + \frac{2}{3}\bar{\phi}\overleftrightarrow{D}_\mu\phi \quad (6)$$

which is the $\theta = 0$ component of the superfield $J_{\alpha\dot{\alpha}}$. Assuming that the matter multiplets are in an irreducible representation of the gauge group, the Konishi operator is the unique renormalizable non-chiral deformation of a critical supersymmetric gauge theory. Its axial vector component $K_\mu(x) = \frac{1}{2}\bar{\psi}_i\gamma_\mu\gamma_5\psi^i + \bar{\phi}_i\overleftrightarrow{D}_\mu\phi^i$ is not conserved classically if the theory contains a superpotential, and it also suffers the Konishi anomaly [10]. Both effects generate an anomalous dimension h , so its scale dimension is $2 + h$.

In our two-loop approximation the anomalous dimension h is only due to the superpotential. We can give a simple argument for this. As we shall see in detail, a correlator like $\langle K_\mu(x)K_\nu(0) \rangle$ maintains its conformal properties to the two-loop order. Conformal symmetry requires that it is proportional to $(\delta_{\mu\nu} - 2x_\mu x_\nu/|x|^2)/|x|^{6+2h}$, so its divergence $\langle \partial K(x)K_\nu(0) \rangle$ is proportional to h . On the other hand ∂K is the sum of two contributions, from the anomaly, proportional to $F\tilde{F}$, and from the superpotential. However, the former only starts contributing at three-loop order, as can be seen by considering the relevant Feynman

diagrams. Hence, only the superpotential term contributes to h . For a similar reason R_μ has no anomalous dimension to our order.

The free field calculations of [7] revealed the presence of the operator $\Sigma_{free}(z) = \bar{\Phi}_i \Phi^i$ in the $J_{\alpha\dot{\alpha}}(z)J_{\beta\dot{\beta}}(z')$ OPE in (1). In the interacting case we *define* (see eq. (7)) the operator which appears in this position as the real superfield $\Sigma(z)$ and refer to its lowest component as $\Sigma(x)$. We can write $\Sigma(z) = \rho(g, Y)K(z)$, since $K(z)$ is the only gauge invariant composite operator of the correct dimension. $\rho(g, Y)$ is a function of the couplings that we determine in lowest nontrivial (two-loop) order. K is the renormalized Konishi operator defined to carry the power μ^h of the renormalization scale, so that ρ is dimensionless.

For simplicity our discussion has assumed that matter multiplets are in an irreducible representation of the gauge group. In the reducible case there is a Konishi superfield K_I for each irreducible component, and the K_I mix under renormalization. Nevertheless, as will be shown elsewhere [11], there is a unique central charge associated with the set of K_I which coincides through two-loop order with c' computed below.

Quantum corrections to c have been computed previously [12], and we shall present them below. To compute corrections to c' and h it proved most convenient to study the OPE's (1) using the $R_\mu(x)$ component of $J_{\alpha\dot{\alpha}}(z)$ and the $K(x)$ (and also the $K_\mu(x)$) components of $K(z)$.

The component OPE's

$$R_\mu(x)R_\nu(y)|_{x \rightarrow y} = \frac{1}{3\pi^4}(\partial_\mu\partial_\nu - \square\delta_{\mu\nu})\frac{c}{(x-y)^4} + \frac{2}{9\pi^2}\Sigma(y)(\partial_\mu\partial_\nu - \delta_{\mu\nu}\square)\frac{1}{(x-y)^{2-h}} + \dots \quad (7)$$

$$\Sigma(x)\Sigma(y)|_{x \rightarrow y} = \frac{1}{16\pi^4}\frac{c'}{(x-y)^{4+2h}} + \dots \quad (8)$$

where $+\dots$ denote less singular terms, precisely define the central charges c , c' , the anomalous dimension h , and the operator $\Sigma(x)$. The numerical factor $1/3\pi^4$ was chosen so that c has the same value in the $R_\mu R_\nu$ OPE as in its conventional definition (2) from the $T_{\mu\nu}T_{\rho\sigma}$ OPE and the curved space trace anomaly. The transverse tensor structure in (7) is valid to two-loop order, even though R_μ is not conserved off criticality due to the chiral anomaly. To prove this, it is sufficient to consider the divergence $\partial R(x)R_\nu(y)|_{x \rightarrow y}$. The anomaly operator equation shows that this OPE is proportional to $g^2 F\tilde{F}(x)R_\nu(y)$, which does not contribute to the right hand side of (7). The effects of nonconservation appear beginning at three-loop order where additional nonconserved tensor structures are present.

We have performed two independent computations of c' and h . The first one involved the study of the connected four-point correlation function $\langle R_\mu(x)R_\nu(y)R_\rho(z)R_\sigma(w) \rangle$ in the asymptotic region where $|x-y|, |z-w| \ll |x-z|, |y-w|$. In this limit we expect the $R_\mu(x)R_\nu(y)$ and $R_\rho(z)R_\sigma(w)$ OPE's to be dominated by the operator $\Sigma(y)$ so that, according to (7)-(8), we should find the asymptotic form

$$\langle R_\mu(x)R_\nu(y)R_\rho(z)R_\sigma(w) \rangle \sim \frac{c'}{324\pi^8}(\partial_\mu\partial_\nu - \delta_{\mu\nu}\square)\frac{1}{(x-y)^{2-h}}$$

$$\times \frac{1}{(x-z)^{4+2h}} (\partial_\rho \partial_\sigma - \delta_{\rho\sigma} \square) \frac{1}{(z-w)^{2-h}}. \quad (9)$$

Note that no scale appears in (9) which is what is expected from a correlator of conserved currents of fixed dimension 3 in a conformal theory.

In perturbation theory, the effect of h is logarithmic, so that we computed the asymptotic form of all one- and two-loop four-point graphs looking for terms of the form

$$\begin{aligned} \langle R(x)R(y)R(z)R(w) \rangle &\sim \frac{1}{(x-y)^4(x-z)^4(z-w)^4} \\ &\cdot \left[N_\chi + ag^2 + a'|Y|^2 + a''|Y|^2 \ln \frac{|x-y||z-w|}{|x-z|^2} \right] \end{aligned} \quad (10)$$

Graphs which contain the indicated asymptotic structure all have $\phi\bar{\phi}$ lines in the intermediate state in the $x-z$ channel, as expected from the relation between $\Sigma(x)$ and $K(x)$. For the superpotential contribution, to avoid some technical problems due to the anomalous dimension of the Konishi operator, we computed the four-point correlator of $J_{\alpha\dot{\alpha}}$ in superspace. The gauge interactions do not contribute to the anomalous dimension and the simplest way to compute their contributions to the correlator (9) was to work in components using the Feynman gauge for gluons. This gauge is convenient, because the one-loop correction to the scalar propagator due to the virtual gluons and gluini vanishes. Regularization by dimensional reduction was used in the intermediate stage of the computation, although divergences cancel in the final contributions to a , a' and a'' . One should note that this method of calculation does not give the proportionality factor $\rho(g, Y)$ between $\Sigma(z)$ and $K(z)$.

Our second method requires the two-loop computation of the two- and three-point correlators $\langle K(x)K(y) \rangle$ and $\langle R_\mu(x)R_\nu(y)K(z) \rangle$. We exploit the well-known fact that in a conformally invariant theory, the conformal symmetry largely, indeed frequently uniquely, fixes the form of two- and three-point correlation functions [13]. The methods of conformal symmetry are applicable here despite the usual lore that the divergences of perturbation theory spoil conformal properties of a theory. Briefly, the reasons are that correlation functions for non-coincident spatial points have no overall divergence, and the only subdivergences which lead to violation of conformal Ward identities correspond to renormalization of the couplings g and Y . Such renormalization subdivergences do not appear until three-loop order in the correlators $\langle R_\mu(x)R_\nu(y)K(z) \rangle$ and $\langle K(x)K(y) \rangle$.

Scale and translation properties are sufficient to determine the form

$$\langle K(x)K(y) \rangle = \frac{A}{16\pi^4(x-y)^{4+2h}} \quad (11)$$

with $A = c'/\rho^2$. The tensor form of $\langle R_\mu(x)R_\nu(y)K(z) \rangle$ can be fixed by the additional requirements of conservation and correct transformation properties under the discrete inversion element of the conformal group, i.e. $x^\mu \rightarrow x'^\mu = x^\mu/x^2$. The result is the conformal tensor

$$\langle R_\mu(x)R_\nu(y)K(z) \rangle = \frac{B}{36\pi^6} \frac{\left(1 - \frac{h}{4}\right) J_{\mu\nu}(x-y) - \frac{1}{2} \left(1 + \frac{h}{2}\right) J_{\mu\rho}(x-z) J_{\rho\nu}(z-y)}{(x-y)^{4-h}(x-z)^{2+h}(y-z)^{2+h}}, \quad (12)$$

where the tensor $J_{\mu\nu}(x) = \delta_{\mu\nu} - 2x_\mu x_\nu / x^2$ is an orthogonal matrix which is (essentially) the Jacobian of the coordinate inversion [14, 15, 13]. In the limit $x \sim y$ one finds the most singular term

$$\langle R_\mu(x) R_\nu(y) K(z) \rangle \sim \frac{B}{72\pi^6(h-2)} \frac{1}{(y-z)^{4+2h}} (\partial_\mu \partial_\nu - \square \delta_{\mu\nu}) \frac{1}{(x-y)^{2-h}}. \quad (13)$$

Use of the OPE's (7)-(8) then immediately yields $c'/\rho = B/(h-2)$. Then, a computation of the indicated two- and three-point correlators gives A , B and h , and determines

$$c'(g, Y) = \frac{B^2}{(h-2)^2 A} \quad , \quad \rho = \frac{B}{(h-2)A}. \quad (14)$$

The first expression allows us to check that c' does not depend on the subtraction scheme. Indeed, scheme dependence manifests itself as the arbitrariness of redefining the renormalized operator K with a finite multiplicative factor of the form $1 + a|Y|^2$. Such a redefinition affects ρ , as well as A and B , but cancels in the formula for c' and in the expression $\Sigma = \rho K$.

We used conformal symmetry methods [15] to facilitate the calculation of the Feynman diagrams. In general these methods are quite straightforward for graphs with virtual scalars and spinors and more difficult for virtual photons [15] because the photon propagator transforms covariantly under a conformal transformation only when accompanied by a gauge transformation. For this reason we studied separately the contribution to c' and ρ from the superpotential and from the gauge interactions of the Lagrangian (5). In the superpotential sector we simply combined conformal methods and those of differential renormalization [16] to give an exact calculation of all two-loop diagrams for the correlators and calculated c' from the quotient (14).

For the gauge sector, in order to take advantage of existing calculations, we considered correlators of the axial vector component of $K(z)$ rather than its scalar component. A discussion similar to that given above applies to the correlators $\langle R_\mu(x) R_\nu(y) K_\rho(z) \rangle$ and $\langle K_\mu(x) K_\nu(y) \rangle$. There are unique conformal invariant tensor forms [14] whose coefficients are the same as A and B in (11) and (12) because of supersymmetry. The two-loop graphs contributing to $\langle R_\mu(x) R_\nu(y) K_\rho(z) \rangle$ lead to integrals that have been evaluated by conformal techniques in previous work on possible radiative corrections to the chiral anomaly in quantum electrodynamics [15] and the standard model [17]. In all these cases the net two-loop contributions to the axial three-point function vanishes, which means that B in the gauge sector of our problem is exactly given by the one-loop value. This computation did not require any regularization because we used a gauge in which the required vertex and self-energy insertions were finite. The correlator $\langle K_\mu(x) K_\nu(y) \rangle$ and the value of A was then obtained from a superspace computation of $\langle K(z) K(z') \rangle$ using both dimensional reduction and differential renormalization which both gave the same result for c' in agreement with the first method. We do not give the result for ρ , for which the order $|Y|^2$ term is scheme dependent, because we have not described the scheme precisely enough for this to be meaningful. The agreement of the two methods of calculation based on the four-point and on the combined three- and two-point correlators provides an excellent cross-check on the fundamental central charge c' .

The calculations described above give the results

$$c' = N_\chi + 2\gamma^i_i \quad , \quad h = \frac{3Y_{ijk}\bar{Y}^{ijk}}{16\pi^2 N_\chi} \quad (15)$$

Quantum corrections to c up to two loops are known from earlier work [12] for a general renormalizable theory of vectors, spinors and scalars in curved space. The results of [12] can be restated for a supersymmetric gauge theory as

$$c = \frac{1}{24} \left(3N_v + N_\chi + N_v \frac{\beta(g)}{g} - \gamma^i_i \right). \quad (16)$$

These results involve the gauge β -function and the anomalous dimension γ^i_j of the chiral superfields which, at the one-loop level are [18]

$$\begin{aligned} 16\pi^2 \beta(g) &= g^3 \left(-3C(G) + \frac{\text{Tr } C(T)}{\dim G} \right), \\ \beta_{ijk} &= Y_{m(ij} \gamma^m_{k)}, \\ 16\pi^2 \gamma^i_j &= \frac{1}{2} Y_{jkm} \bar{Y}^{ikm} - 2g^2 C(T)^i_j, \\ C(G) \delta^{ab} &= f^{acd} f^{bcd}, \\ C(T)^i_j &= (T^a T^a)^i_j. \end{aligned} \quad (17)$$

We observe now that the two-loop corrections to c and c' vanish if the conditions $\beta(g) = 0$ and $\gamma^i_i = 0$ are satisfied, and these conditions mean that the theory is conformal invariant to one-loop order. Conversely, if both c and c' are uncorrected (not just c), then the theory is conformal. However, even under these conditions, for nonvanishing superpotential the anomalous dimension h is not zero, even for such superconformal theories as N=4 Yang-Mills theory.

It is known that to all orders in the couplings g, Y there is a fixed surface of the renormalization group flow provided that the matter representation is chosen so that $3 \dim G C(G) - \text{Tr } C(T) = 0$ and g, Y are such that $\gamma^i_j = 0$. This can be seen from the NSVZ relation [19]

$$16\pi^2 \beta^{NSVZ} = -g^3 \frac{3C(G) \dim G - \text{Tr } [(1 - 2\gamma) C(T)]}{\dim G \left(1 - \frac{g^2}{8\pi^2} C(G) \right)},$$

and the relation between β_{ijk} and the anomalous dimension. Therefore, there exists a space of continuously connected conformal invariant theories and we have found evidence that the two central charges c and c' are universal, i.e. that c and c' are constant, independent of the couplings, on this space. We call such quantities SCFT₄ invariants. This universality property is the four-dimensional analogue of the well known fact that the Virasoro central charge of a two-dimensional conformal theory is invariant under marginal deformations. The other important information that we have obtained is that in four dimensions the OPE's do contain a third quantity, namely h , that is not an SCFT₄ invariant. h tells us that continuously connected SCFT₄'s are indeed inequivalent. These results were anticipated in [7] and we regard them as the essential features of superconformal (and conformal) field theories in four dimensions.

Possible c -theorems.

At this point we discuss the search for a c -function in four-dimensional quantum field theory, a function of the couplings of the theory which must decrease along renormalization group flows toward the infrared reflecting the continuous "integrating out" of degrees of freedom and which is stationary at fixed points of the flow. Since there are two well-defined central charges in SCFT₄, we can consider c and c' , which according to (3) and (4) measure, respectively, the total number of degrees of freedom and the total number of scalar multiplets. Or we can examine the quantity $N_v^{eff} = 8c - c'/3$ which is a measure of the effective number of vector multiplets. A c -theorem for one or more of these quantities c , N_v^{eff} or c' , could allow a test of phenomena such as confinement, chiral and supersymmetry breaking and the Higgs mechanism. There are known non-supersymmetric examples [5] of both increasing and decreasing behavior of c along RG flows, so that c is not a good c -function, but the situation could be better in supersymmetry [20]. Our analysis will show that none of the quantities c , c' and N_v^{eff} satisfies the desiderata of a c -function for the full class of $N=1$ supersymmetric gauge theories with superpotential. Yet, c' seems to have an interesting behaviour for the subclass of asymptotically free theories.

We begin by examining theories with vanishing superpotential ($Y = 0$). In this case we obtain from (16)

$$c = \frac{1}{24} \left[3N_v + N_\chi + \frac{3g^2}{16\pi^2} (-C(G)N_v + \text{Tr } C(T)) \right]. \quad (18)$$

From (17) we learn that the theory is UV free if $\text{Tr } C(T) < 3C(G)N_v$, so that we can consider the change in g as we move down from the UV fixed point. We see that c decreases if $\text{Tr } C(T) < C(G)N_v$, but increases outside of this range. For $SU(N_c)$ supersymmetric QCD with N_f fundamental quarks and their anti-quarks, this equality becomes $N_f < N_c$ which is a small part of the range of asymptotic freedom $N_f < 3N_c$ and, except for $N_f = 0$, exactly the range for which non-perturbative considerations indicate that there is no ground state [21]. Remarks similar to those above hold for N_v^{eff} , but the range in which this quantity decreases, namely $N_f < 3N_c/7$, is smaller than for c . So it is only for the case of pure supersymmetric Yang-Mills theory without chiral matter that we obtain a decreasing c -function.

We can also restrict to supersymmetric gauge theories for which the one-loop β -function vanishes and examine the behavior of our c -function candidates on flows in the neighborhood of the fixed surface discussed above. This surface is known [22] to be infrared attractive on both sides. But the radiative corrections to c , c' and N_v^{eff} are linear in γ_j^i , so that each of these quantities decreases along flows on one side of the fixed surface, but increases along flows on the other side.

We now take a closer look at

$$c' = N_\chi - \frac{1}{4\pi^2} \left[g^2 \text{Tr } C(T) - \frac{1}{4} Y_{ijk} \bar{Y}^{ijk} \right]. \quad (19)$$

We first note that when $Y = 0$ c' decreases from the ultraviolet in an asymptotically free gauge theory. It is interesting to see in detail what happens when g and Y are both nonzero. The

one-loop flow equations suggested by (17) read

$$\begin{aligned}\frac{dy}{dt} &= by(qy^2 - rg^2), \\ \frac{dg}{dt} &= -bg^3,\end{aligned}\tag{20}$$

where $b > 0$ in an asymptotically free theory. Dividing and writing $y = gz$ we obtain

$$g\frac{dz}{dg} = (r-1)z - qz^3.\tag{21}$$

If the initial value of z satisfies $z^2 < \frac{r-1}{q}$, then there is a solution for $y(g)$ which behaves at small g as $y \sim g^r$. So the Yukawa coupling can be asymptotically free in the presence of a non-abelian gauge coupling¹. If $r > 1$ then $y \rightarrow 0$ faster than g and c' is a decreasing function as we move away from the ultraviolet fixed point in the space of g, Y .

The intriguing fact that c' appears to be a good c -function near the ultraviolet fixed point of asymptotically free gauge theories motivates us to test its infrared behavior by extending Bastianelli's analysis [20] to c' . The infrared fixed point of $SU(N_c)$ supersymmetric gauge theory with $N_c + 2 \leq N_f < 3N_c$ quark flavors is conjectured to be described by the magnetic dual theory with gauge group $SU(N_f - N_c)$ with N_f quarks, N_f antiquarks and N_f^2 mesons. In the range $N_c + 2 \leq N_f < 3N_c/2$ the magnetic theory is infrared free, so the free field values of c and c' may be used. There are thus $c'_{IR} = 2N_f(N_f - N_c) + N_f^2$ chiral multiplets in the magnetic theory and $c'_{UV} = 2N_fN_c$ in the electric theory. So one has $c'_{UV} - c'_{IR} = N_f(4N_c - 3N_f)$, which is positive only in part of the range for which the non-perturbative description in terms of a free dual theory is valid. Thus our attempted interpretation of c' as a c -function is problematic when the Seiberg scenario of non-perturbative dynamics is considered.

It is worth stressing again the physical relevance of quantities that count the various kinds of degrees of freedom separately, in particular the quantity N_v^{eff} that measures, in some sense, the size of the effective gauge symmetry group of the theory. Even if these quantities do not satisfy general c -theorems, it is still conceivable that they behave nicely in physical models.

Conclusions.

We summarize our results: for general supersymmetric theories, the classical central charges c, c' and the dimension of the operator Σ receive quantum corrections. When the conditions for superconformal invariance are satisfied the corrections to the central charges vanish. This suggests that c and c' are invariant under continuous deformations of superconformal theories and generalizes the analogous property of two-dimensional quantum field theory. However, in four dimensions the operator Σ appears in the OPE's of the supercurrent even at the conformal point, with nonvanishing anomalous dimension h whenever a superpotential (or Konishi anomaly) is present. Our N=1 component and superspace calculations are also valid for extended N=2 and

¹ These conditions are satisfied in a simple model with $G = SU(N)$, a matter superfield $\phi^a(z)$ in the adjoint representation and superpotential involving the $SU(N)$ d -symbol, in which case $r = 3$.

N=4 models, because c' and h are scheme independent to the order considered. In particular the stress-tensor OPE in the conformally invariant N=4 theory contains components of the Konishi operator with nonvanishing anomalous dimension; extended supersymmetry does not simplify the form of the OPE's. We believe that these are the essential features of the operator product algebra of superconformal field theories in four dimensions. (In [23] a form of the TT OPE for the $N = 4$ theory is presented containing only integer powers of the distance, which conflicts with our result.)

We have also examined the possibility of establishing a c -theorem in terms of the quantity c which counts the total number of multiplets, c' which counts the number of chiral multiplets, or the weighted difference N_v^{eff} which counts the number of vector multiplets. We have found that none of them is a suitable candidate in general, although under some restrictive, and possibly physically relevant, conditions they might play such a role.

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